Integrable generalised spin ladder models

Angela Foerster[1], Jon Links[1,2] and Arlei Prestes Tonel[1]

[1] *Instituto de Física da UFRGS*
Av. Bento Gonçalves 9500, Porto Alegre, RS - Brazil

[2] *Department of Mathematics*
University of Queensland, Queensland, 4072, Australia

Abstract

We present two new integrable spin ladder models which posses three general free parameters besides the rung coupling J. Wang's systems based on the $SU(4)$ and $SU(3|1)$ symmetries can be obtained as special cases. The models are exactly solvable by means of the Bethe ansatz method.

Recently there has been a great interest on spin ladder systems from both theoretical and experimental point of view for their relevance to some quasi-one dimensional materials, which under hole doping may exhibit superconductivity [1]. These systems are reasonably well approximated by Heisenberg ladders, which take into account only couplings along the legs and the rungs. Although these systems are not exactly solvable, a variety of solvable ladder models have been found [2], [3], [4]. Of particular interest is a general 2-leg spin ladder system with biquadratic interactions proposed by Wang [5], which by suitable choices of the interchain and interrung coupling originates two integrable spin ladder models based on the $SU(4)$ and $SU(3/1)$ symmetries. In these cases the rung interactions appear as chemical potentials which break the underlying symmetries of the models. Subsequently other generalised integrable spin ladders have been proposed in the literature [6, 7, 8, 9, 10, 11]. In these cases, no (or few) free parameters are present due to the strict conditions of integrability.

The purpose of this paper is to present two new integrable generalized spin ladders with three extra parameters without violating integrability. These models are exactly solvable by the Bethe ansatz and they reduce to Wang's models [5] for a special limit of these extra parameters.

Let us begin by introducing the first generalised spin ladder model, whose Hamiltonian reads

$$H^{(1)} = \sum_{j=1}^{L} \Big[h_{j,j+1} + \tfrac{1}{2} J \, (\vec{\sigma_j}.\vec{\tau_j} - 1) \Big] \tag{1}$$

where

$$\begin{aligned}
h_{j,j+1} \;&= \sigma_j^+ \sigma_{j+1}^- \left[\frac{t_1^{-2}}{4}(1+\tau_j^z)(1+\tau_{j+1}^z) + \frac{t_2^2}{4}(1-\tau_j^z)(1-\tau_{j+1}^z) + t_3^2\, \tau_j^+ \tau_{j+1}^- + \tau_j^- \tau_{j+1}^+ \right] \\
&+ \sigma_j^- \sigma_{j+1}^+ \left[\frac{t_1^2}{4}(1+\tau_j^z)(1+\tau_{j+1}^z) + \frac{t_2^{-2}}{4}(1-\tau_j^z)(1-\tau_{j+1}^z) + \tau_j^+ \tau_{j+1}^- + t_3^{-2}\, \tau_j^- \tau_{j+1}^+ \right] \\
&+ \tfrac{1}{4}(1+\sigma_j^z)(1+\sigma_{j+1}^z) \left[\tfrac{1}{2}(1+\tau_j^z \tau_{j+1}^z) + t_1^{-2}\, \tau_j^+ \tau_{j+1}^- + t_1^2\, \tau_j^- \tau_{j+1}^+ \right] \\
&+ \tfrac{1}{4}(1-\sigma_j^z)(1-\sigma_{j+1}^z) \left[\tfrac{1}{2}(1+\tau_j^z \tau_{j+1}^z) + t_2^2\, \tau_j^+ \tau_{j+1}^- + t_2^{-2} \tau_j^- \tau_{j+1}^+ \right].
\end{aligned}$$

Above $\vec{\sigma_j}$ and $\vec{\tau_j}$ are Pauli matrices acting on site j of the upper and lower legs, respectively, J is the strength of the rung coupling and t_1, t_2, t_3 are general independent parameters. L is the number of rungs and periodic boundary conditions are imposed. By setting $t_1, t_2, t_3 \to 1$ in equation (1), Wang's model based on the $SU(4)$ symmetry [1][5] can be recovered.

The integrability of this model can be shown by the fact that it can be mapped to the Hamiltonian below, which can be derived from an $R-$matrix obeying the Yang-Baxter algebra for $J = 0$, while for $J \neq 0$ the rung interactions take the form of a chemical potential term.

$$\hat{H}^{(1)} = \sum_{j=1}^{L} \Big[\hat{h}_{j,j+1} - 2J \, X_j^{00} \Big] \tag{2}$$

[1]strictly speaking, it is SU(4) in the absence of the rung coupling

where

$$
\begin{aligned}
\hat{h}_{j,j+1} \; &= \sum_{\alpha=0}^{3} X_j^{\alpha\alpha} X_{j+1}^{\alpha\alpha} + X_j^{20} X_{j+1}^{02} + X_j^{02} X_{j+1}^{20} \\
&+ t_1^2 \Big(X_j^{10} X_{j+1}^{01} + X_j^{12} X_{j+1}^{21} \Big) + t_2^2 \Big(X_j^{30} X_{j+1}^{03} + X_j^{32} X_{j+1}^{23} \Big) + t_3^2 X_j^{31} X_{j+1}^{13} \\
&+ t_1^{-2} \Big(X_j^{01} X_{j+1}^{10} + X_j^{21} X_{j+1}^{12} \Big) + t_2^{-2} \Big(X_j^{03} X_{j+1}^{30} + X_j^{23} X_{j+1}^{32} \Big) + t_3^{-2} X_j^{13} X_{j+1}^{31}.
\end{aligned}
$$

Above $X_j^{\alpha\beta} = |\alpha_j >< \beta_j|$ are the Hubbard operators with $|\alpha_j >$ the orthogonalised eigenstates of the local operator $\vec{\sigma_j}.\vec{\tau_j}$, as in Wang's case [5].

The following R-matrix

$$
R = \left(\begin{array}{cccc|cccc|cccc|cccc}
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t_1^{-2}b & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t_2^{-2}b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\
\hline
0 & c & 0 & 0 & t_1^2 b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t_1^2 b & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t_3^{-2}b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\
\hline
0 & 0 & c & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & t_1^{-2}b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_2^{-2}b & 0 & 0 & c & 0 \\
\hline
0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_2^2 b & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & t_3^2 b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & t_2^2 b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a
\end{array} \right), \tag{3}
$$

with

$$a = x + 1 \quad , b = x, \quad c = 1,$$

obeys the Yang-Baxter algebra

$$R_{12}(x-y) R_{13}(x) R_{23}(y) = R_{23}(y) R_{13}(x) R_{12}(x-y) \tag{4}$$

and originates the Hamiltonian (2) for $J = 0$ by the standard procedure

$$\hat{h}_{j,j+1} = P \frac{d}{dx} R(x)|_{x=0},$$

where P is the permutation operator.

The model can be solved exactly by the Bethe ansatz method and the Bethe ansatz equations read

$$t_1^{2(L-M_3)} t_2^{2M_3} t_3^{-2M_3} \left(\frac{\lambda_j - i/2}{\lambda_j + i/2} \right)^L \;\; = \;\; \prod_{l \neq j}^{M_1} \frac{\lambda_j - \lambda_l - i}{\lambda_j - \lambda_l + i} \prod_{\alpha=1}^{M_2} \frac{\lambda_j - \mu_\alpha + i/2}{\lambda_j - \mu_\alpha - i/2}$$

$$
\begin{aligned}
t_1^{2(L-M_3)} t_2^{2M_3} t_3^{-2M_3} \prod_{\beta\neq\alpha}^{M_2} \frac{\mu_\alpha-\mu_\beta-i}{\mu_\alpha-\mu_\beta+i} &= \prod_{j=1}^{M_1} \frac{\mu_\alpha-\lambda_j-i/2}{\mu_\alpha-\lambda_j+i/2} \prod_{\delta=1}^{M_3} \frac{\mu_\alpha-\nu_\delta-i/2}{\mu_\alpha-\nu_\delta+i/2} \qquad (5)\\
t_1^{2(M_2-M_1)} t_2^{-2(L-M_1+M_2)} t_3^{-2(M_1-M_2} \prod_{\gamma\neq\delta}^{M_3} \frac{\nu_\delta-\nu_\gamma-i}{\nu_\delta-\nu_\gamma+i} &= \prod_{\alpha=1}^{M_2} \frac{\nu_\delta-\mu_\alpha-i/2}{\nu_\delta-\mu_\alpha+i/2}
\end{aligned}
$$

The energy eigenvalues of the Hamiltonian (2) are given by

$$
E = -\sum_{j=1}^{M_1} \left(\frac{1}{\lambda_j^2+1/4} - 2J \right) + \frac{3}{4}\left(1-2J\right) L \qquad (6)
$$

where λ_j are solutions to the Bethe ansatz equations (5).

Now let us introduce a second integrable spin ladder model with three extra parameters, whose Hamiltonian reads

$$
\mathcal{H} = \sum_{j=1}^{L} \left[k_{j,j+1} - \frac{1}{4}(1-2J)(\vec{\sigma_j}.\vec{\tau_j}-1) \right] \qquad (7)
$$

where

$$
k_{j,j+1} = h_{j,j+1} - \frac{1}{8}(\vec{\sigma_j}.\vec{\tau_j})(\vec{\sigma_{j+1}}.\vec{\tau_{j+1}}) + \frac{1}{4}\left(\vec{\sigma_j}.\vec{\tau_j}-1\right) \qquad (8)
$$

and $h_{j,j+1}$ is given by eq. (1).

The solvability of Hamiltonian above lies in the fact that it can be mapped, as before, to a Hamiltonian which can be derived for an R-matrix satisfying the Yang-Baxter algebra for $J=0$, while for $J\neq 0$ the rung interactions take the form of a chemical potential term

$$
\hat{\mathcal{H}} = \sum_{j=1}^{L} \left[\hat{k}_{j,j+1} + (1-2J) X_j^{00} \right] \qquad (9)
$$

where

$$
\begin{aligned}
\hat{k}_{j,j+1} &= \sum_{\alpha=0}^{3} X_j^{\alpha\alpha} X_{j+1}^{\alpha\alpha} - 2\, X_j^{00} X_{j+1}^{00} + X_j^{20} X_{j+1}^{02} + X_j^{02} X_{j+1}^{20} \\
&+ t_1^2 \left(X_j^{10} X_{j+1}^{01} + X_j^{12} X_{j+1}^{21} \right) + t_2^2 \left(X_j^{30} X_{j+1}^{03} + X_j^{32} X_{j+1}^{23} \right) + t_3^2 X_j^{31} X_{j+1}^{13} \\
&+ t_1^{-2} \left(X_j^{01} X_{j+1}^{10} + X_j^{21} X_{j+1}^{12} \right) + t_2^{-2} \left(X_j^{03} X_{j+1}^{30} + X_j^{23} X_{j+1}^{32} \right) + t_3^{-2} X_j^{13} X_{j+1}^{31}. \qquad (10)
\end{aligned}
$$

For $J=0$ the model is derived by standard methods from the following $\mathcal{R}$-matrix

$$\mathcal{R} = \left(\begin{array}{cccc|cccc|cccc|cccc}
w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t_1^{-2}b & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t_2^{-2}b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\
\hline
0 & c & 0 & 0 & t_1^{2}b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t_1^{2}b & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t_3^{-2}b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\
\hline
0 & 0 & c & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & t_1^{-2}b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_2^{-2}b & 0 & 0 & c & 0 \\
\hline
0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_2^{2}b & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & t_3^{2}b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & t_2^{2}b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a
\end{array} \right), \tag{11}$$

with

$$a = x+1 \quad , b = x, \quad c = 1 \quad , w = -x+1,$$

which obeys the Yang-Baxter algebra. The above Hamiltonian has a similar algebraic structure as that of an SU(3/1) supersymmetric t-J model. Using the algebraic nested Bethe ansatz method this model can be solved and the Bethe ansatz equation are given by

$$\begin{aligned}
t_1^{2(L-M_3)} t_2^{2M_3} t_3^{-2M_3} \left(\frac{\lambda_j - i/2}{\lambda_j + i/2} \right)^L &= \prod_{\alpha=1}^{M_2} \frac{\lambda_j - \mu_\alpha - i/2}{\lambda_j - \mu_\alpha + i/2} \\
t_1^{2(L-M_3)} t_2^{2M_3} t_3^{-2M_3} \prod_{\beta \neq \alpha}^{M_2} \frac{\mu_\alpha - \mu_\beta - i}{\mu_\alpha - \mu_\beta + i} &= \prod_{j=1}^{M_1} \frac{\mu_\alpha - \lambda_j - i/2}{\mu_\alpha - \lambda_j + i/2} \prod_{\delta=1}^{M_3} \frac{\mu_\alpha - \nu_\delta - i/2}{\mu_\alpha - \nu_\delta + i/2} \\
t_1^{2(M_2-M_1)} t_2^{-2(L-M_1+M_2)} t_3^{-2(M_1-M_2)} \prod_{\gamma \neq \delta}^{M_3} \frac{\nu_\delta - \nu_\gamma - i}{\nu_\delta - \nu_\gamma + i} &= \prod_{\alpha=1}^{M_2} \frac{\nu_\delta - \mu_\alpha - i/2}{\nu_\delta - \mu_\alpha + i/2}
\end{aligned} \tag{12}$$

(13)

The eigenenergy of the Hamiltonian (9) is given by

$$E = \sum_{j=1}^{M_1} \Big(\frac{1}{\lambda_j^2 + 1/4} + 2J - 1 \Big) - 2JL \tag{14}$$

above λ_j are solutions of Bethe-ansatz equations (12).

To summarize, we have introduced a new generalization of Wang's spin ladder models based on the $SU(4)$ and $SU(3|1)$ symmetries. This was achieved by introducing three extra parameters into the system without violating integrability. The Bethe ansatz equations as well as the energy expressions of the models were presented. The physics of the

integrable models presented here is expected to be of interest, since the presence of these extra parameters may turn the phase diagram of the models much richer.

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